Models for areal data

Lecture 19

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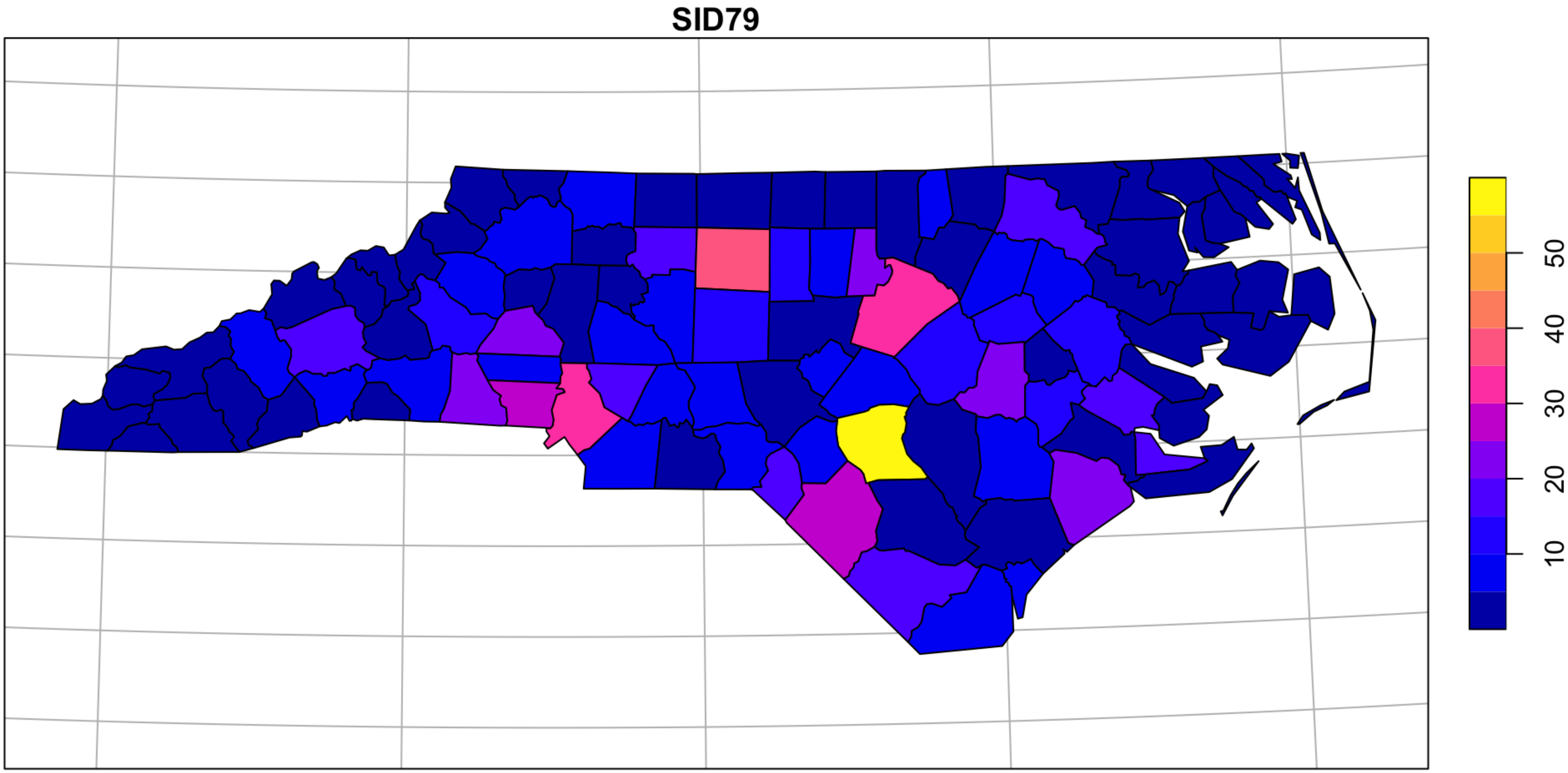
areal / lattice data

Example - NC SIDS



Adjacency Matrix

```
1*st_touches(nc[1:12,], sparse=FALSE)
```

```
      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10] [,11] [,12]
 [1,]    0    1    0    0    0    0    0    0    0     0     0     0
 [2,]    1    0    1    0    0    0    0    0    0     0     0     0
 [3,]    0    1    0    0    0    0    0    0    0     1     0     0
 [4,]    0    0    0    0    0    0    1    0    0     0     0     0
 [5,]    0    0    0    0    0    1    0    0    1     0     0     0
 [6,]    0    0    0    0    1    0    0    1    0     0     0     0
 [7,]    0    0    0    1    0    0    0    1    0     0     0     0
 [8,]    0    0    0    0    0    1    1    0    0     0     0     0
 [9,]    0    0    0    0    1    0    0    0    0     0     0     0
[10,]    0    0    1    0    0    0    0    0    0     0     0     1
[11,]    0    0    0    0    0    0    0    0    0     0     0     1
```

Normalized spatial weight matrix

```r
normalize_weights = function(w) {
  w = 1*w
  diag(w) = 0
  rs = rowSums(w)
  rs[rs == 0] = 1
  w/rs
}
normalize_weights( st_touches(nc[1:12,], sparse=FALSE) )
```

```
      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10] [,11] [,12]
 [1,]  0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0   0.0   0.0   0.0
 [2,]  0.5  0.0  0.5  0.0  0.0  0.0  0.0  0.0  0.0   0.0   0.0   0.0
 [3,]  0.0  0.5  0.0  0.0  0.0  0.0  0.0  0.0  0.0   0.5   0.0   0.0
 [4,]  0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0   0.0   0.0   0.0
 [5,]  0.0  0.0  0.0  0.0  0.0  0.5  0.0  0.0  0.5   0.0   0.0   0.0
 [6,]  0.0  0.0  0.0  0.0  0.5  0.0  0.0  0.5  0.0   0.0   0.0   0.0
 [7,]  0.0  0.0  0.0  0.5  0.0  0.0  0.0  0.5  0.0   0.0   0.0   0.0
 [8,]  0.0  0.0  0.0  0.0  0.0  0.5  0.5  0.0  0.0   0.0   0.0   0.0
 [9,]  0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0   0.0   0.0   0.0
[10,]  0.0  0.0  0.5  0.0  0.0  0.0  0.0  0.0  0.0   0.0   0.0   0.5
[11,]  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0   0.0   0.0   1.0
[12,]  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0   0.5   0.5   0.0
```

EDA - Moran's I

If we have observations at n spatial locations $(s_1, \ldots s_n)$

$$I = \frac{n}{\sum_{i=1}^n \sum_{j=1}^n w_{ij}} \frac{\sum_{i=1}^n \sum_{j=1}^n w_{ij} \big(y(s_i) - \bar{y}\big)\big(y(s_j) - \bar{y}\big)}{\sum_{i=1}^n \big(y(s_i) - \bar{y}\big)^2}$$

where $\boldsymbol{w}$ is a spatial weights matrix.

Some properties of Moran's I when there is no spatial autocorrelation / dependence:

- $E(I) = -1/(n-1)$
- $Var(I) = (\text{Something ugly but closed form}) - E(I)^2$
- $\lim_{n\to\infty} \frac{I - E(I)}{\sqrt{Var(I)}} \sim N(0,1)$ (via the CLT)

NC SIDS & Moran's I

Lets start by using a normalized spatial weight matrix for w (basedd on shared county borders).

```r
morans_I = function(y, w) {
  w = normalize_weights(w)
  n = length(y)
  num = sum(w * (y-mean(y)) %*% t(y-mean(y)))
  denom = sum( (y-mean(y))^2 )
  (n/sum(w)) * (num/denom)
}

w = st_touches(nc, sparse=FALSE)
morans_I(y = nc$SID74, w)
```

```
[1] 0.1477405
```

```r
ape::Moran.I(nc$SID74, weight = w) %>% str()
```

```
List of 4
 $ observed: num 0.148
 $ expected: num -0.0101
 $ sd      : num 0.0627
 $ p.value : num 0.0118
```

EDA - Geary's C

Like Moran's I, if we have observations at n spatial locations $(s_1, \ldots s_n)$

$$C = \frac{n-1}{2\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}} \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\big(y(s_i) - y(s_j)\big)^2}{\sum_{i=1}^{n}\big(y(s_i) - \bar{y}\big)^2}$$

where $\boldsymbol{w}$ is a spatial weights matrix.

Some properties of Geary's C:

- $0 < C < 2$
  - If $C \approx 1$ then no spatial autocorrelation
  - If $C > 1$ then negative spatial autocorrelation
  - If $C < 1$ then positive spatial autocorrelation
- Geary's C is inversely related to Moran's I

NC SIDS & Geary's C

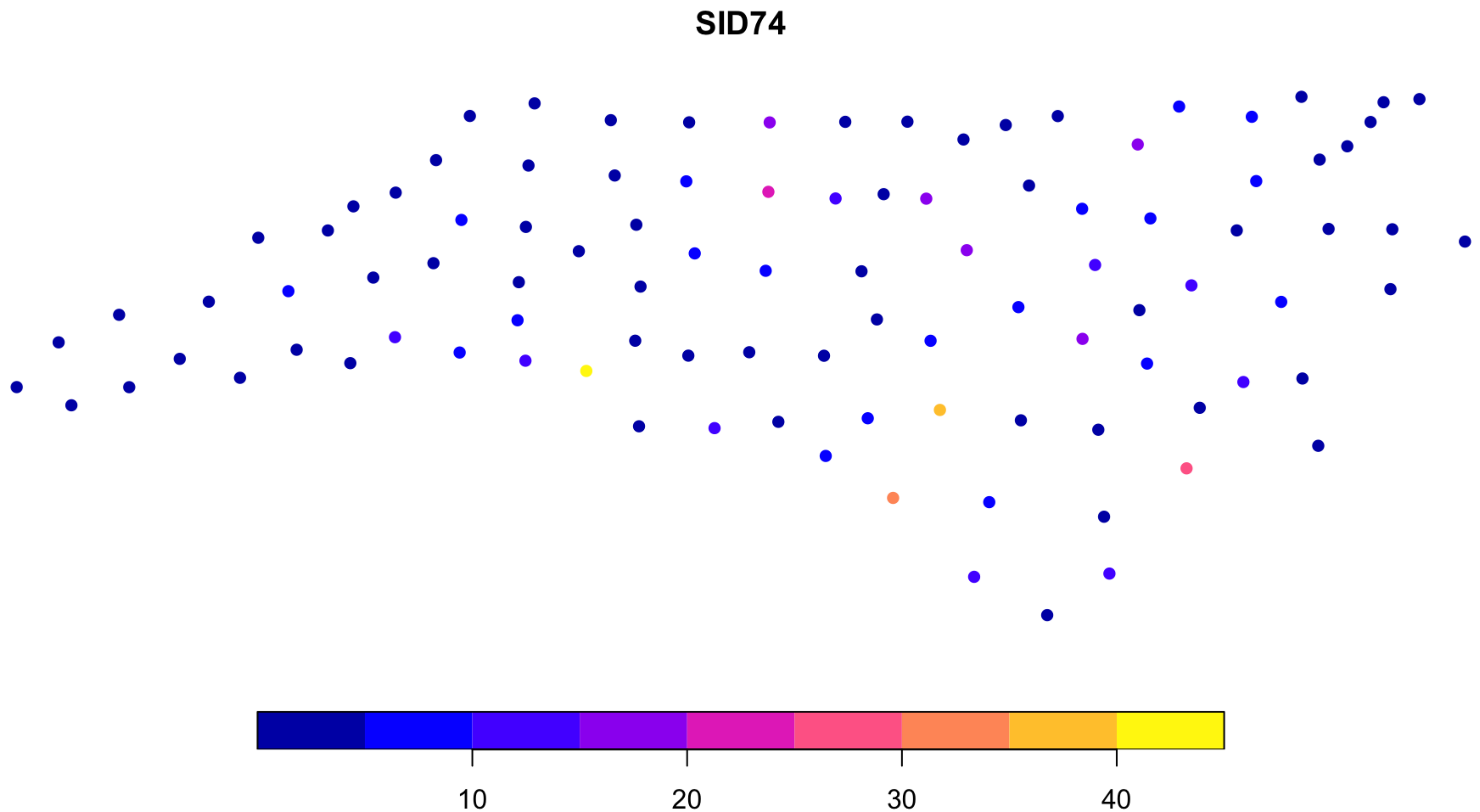
Again using an normalized adjacency matrix for w (shared county borders).

```
gearys_C = function(y, w) {
  w = normalize_weights(w)

  n = length(y)
  y_i = y %*% t(rep(1,n))
  y_j = t(y_i)
  ((n-1)/(2*sum(w))) * (sum(w * (y_i-y_j)^2) / sum( (y - mean(y))^2 ))
}

w = 1*st_touches(nc, sparse=FALSE)
gearys_C(y = nc$SID74, w = w)
```

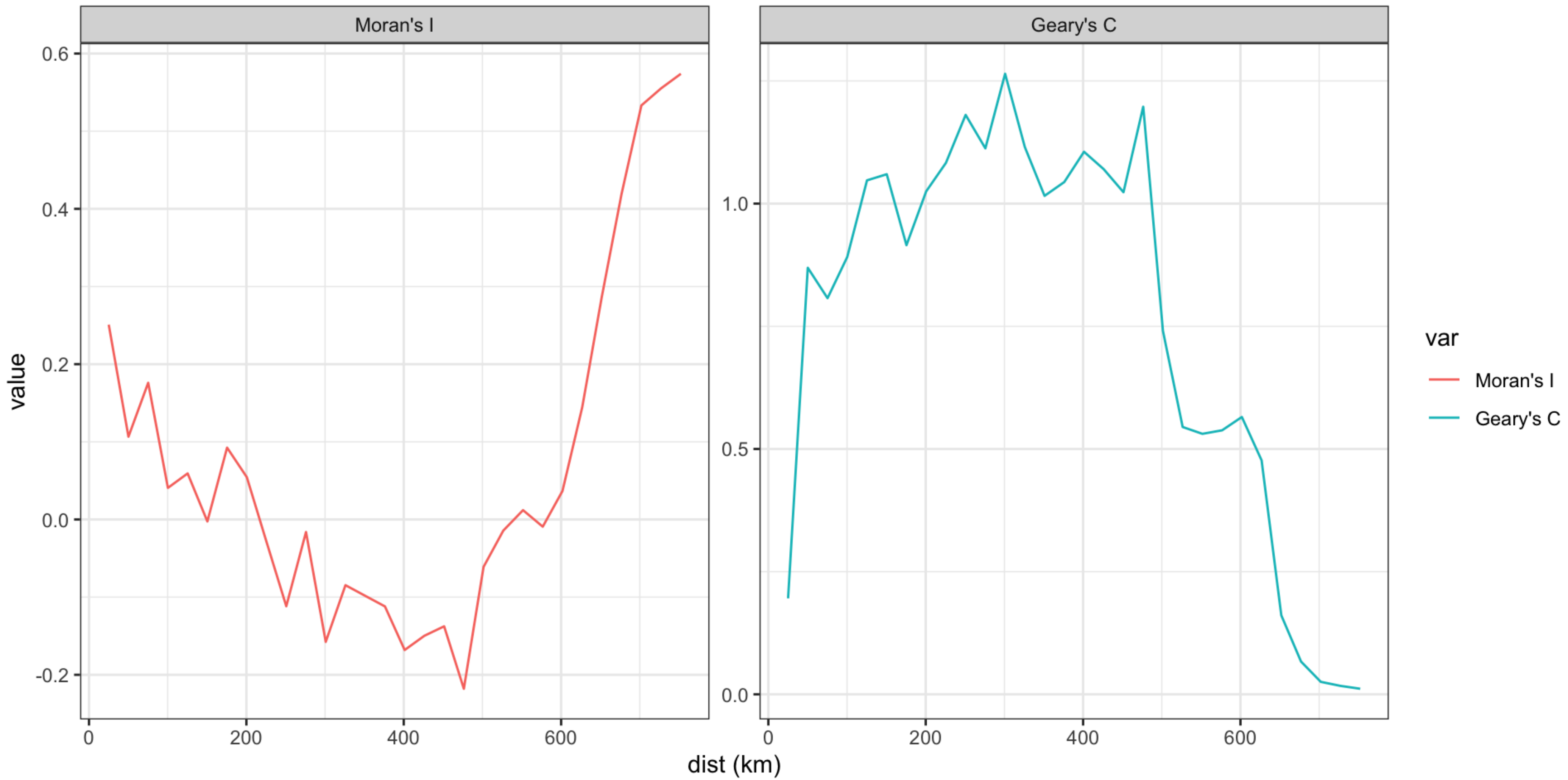
```
[1] 0.8438767
```

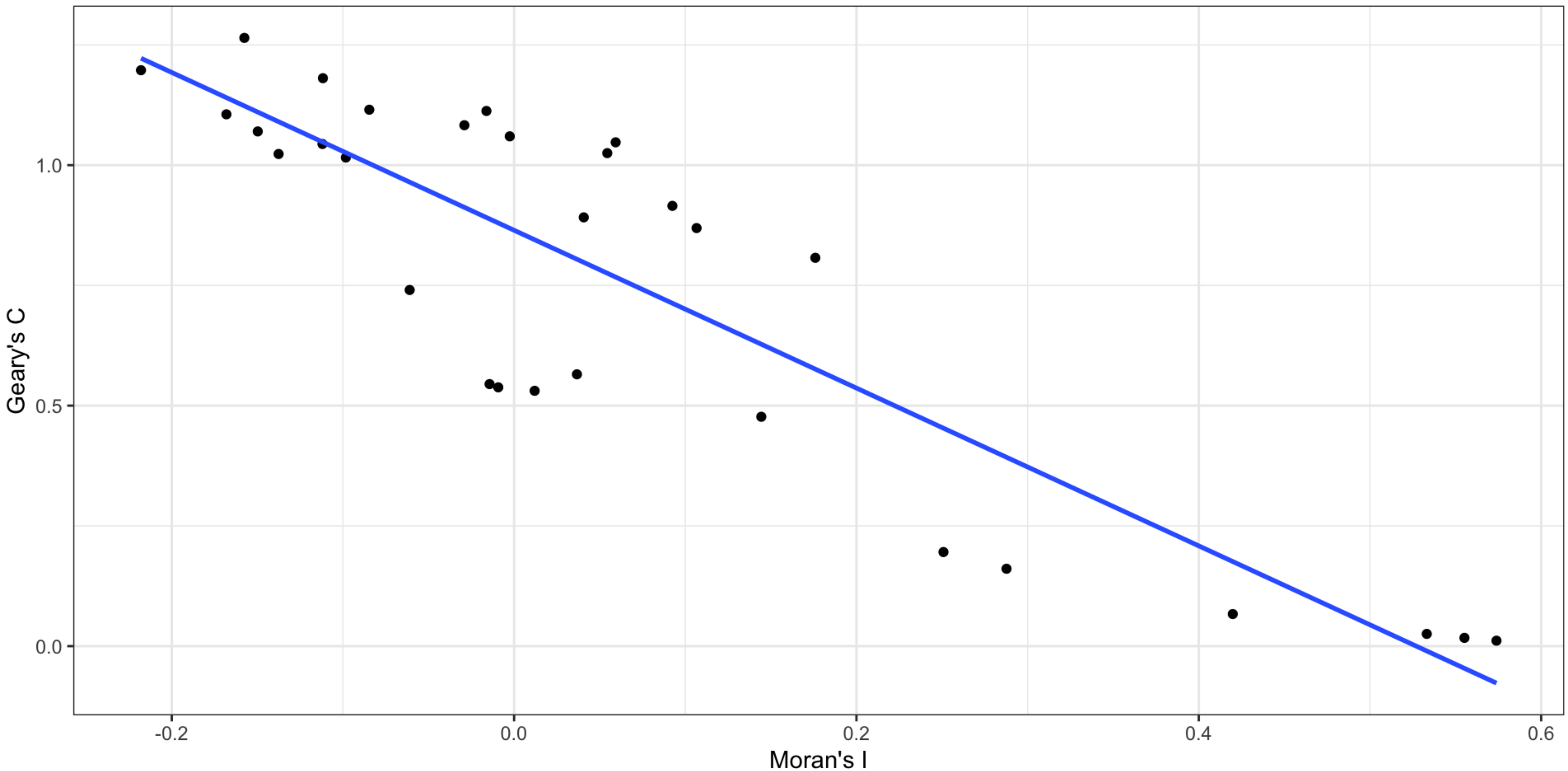
Spatial Correlogram

```r
nc_pt = st_centroid(nc)
plot(nc_pt[,"SID74"], pch=16)
```



Here we are defining adjacency based on the centroid being within a given distance,




Geary's C
1.0
0.5
0.0
-0.2
0.0
0.2
0.4
0.6
Moran's I

Autoregressive Models

AR Models - Time

Lets return to the simplest case, an $\text{AR}(1)$ process

$$y_t = \delta + \phi \, y_{t-1} + w_t$$

where $w_t \sim N(0, \sigma_w^2)$ and $|\phi| < 1$, then

$$E(y_t) = \frac{\delta}{1-\phi}$$

$$Var(y_t) = \frac{\sigma^2}{1-\phi}$$

$$\rho(h) = \phi^h$$

$$\gamma(h) = \phi^h \frac{\sigma^2}{1-\phi}$$

AR Models - Time - Joint Distribution

Previously we saw that an $\text{AR}(1)$ model can be represented using a multivariate normal distribution

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \sim N\left(\frac{\delta}{1-\phi} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},\ \frac{\sigma^2}{1-\phi} \begin{pmatrix} 1 & \phi & \cdots & \phi^{n-1} \\ \phi & 1 & \cdots & \phi^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \phi^{n-1} & \phi^{n-2} & \cdots & 1 \end{pmatrix} \right)$$

In writing down the likelihood we also saw that an $\text{AR}(1)$ is 1st order Markovian,

$$\begin{aligned} f(y_1, \ldots, y_n) &= f(y_1)\, f(y_2 \mid y_1)\, f(y_3 \mid y_2, y_1) \cdots f(y_n \mid y_{n-1}, y_{n-2}, \ldots, y_1) \\ &= f(y_1)\, f(y_2 \mid y_1)\, f(y_3 \mid y_2) \cdots f(y_n \mid y_{n-1}) \end{aligned}$$

Alternative Definitions for y_t

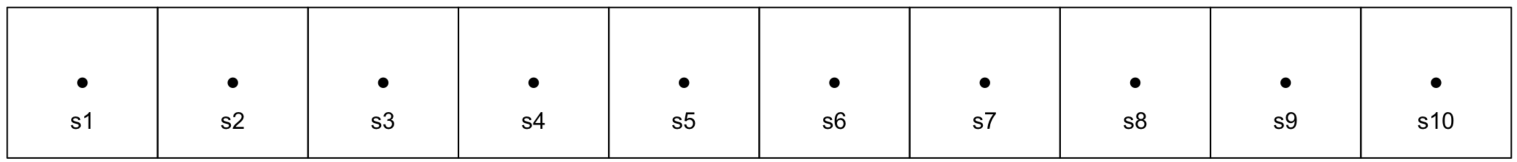
$$y_t = \delta + \phi\, y_{t-1} + w_t$$

vs.

$$y_t \mid y_{t-1} \sim N(\delta + \phi\, y_{t-1},\ \sigma^2)$$

In the case of time, both of these definitions result in the same multivariate distribution for $\boldsymbol{y}$ given on the previous slide.

AR in Space



Even in the simplest spatial case there is no clear / unique ordering,

$$
\begin{aligned}
f\big(y(s_1), \ldots, y(s_{10})\big) &= f\big(y(s_1)\big)\, f\big(y(s_2) \mid y(s_1)\big) \;\cdots\; f\big(y(s_{10} \mid y(s_9), y(s_8), \ldots, y(s_1)\big) \\
&= f\big(y(s_{10})\big)\, f\big(y(s_9) \mid y(s_{10})\big) \;\cdots\; f\big(y(s_1 \mid y(s_2), y(s_3), \ldots, y(s_{10})\big) \\
&= \;?
\end{aligned}
$$

Instead we need to think about things in terms of their neighbors / neighborhoods. We define $N(s_i)$ to be the set of neighbors of location s_i.

Defining the Spatial AR model

Here we will consider a simple average of neighboring observations, just like with the temporal AR model we have two options in terms of defining the autoregressive process,

- Simultaneous Autogressve (SAR)

$$y(s) = \delta + \phi \frac{1}{|N(s)|} \sum_{s' \in N(s)} y(s') + N(0, \sigma^2)$$

- Conditional Autoregressive (CAR)

$$y(s) \mid y(-s) \sim N\left(\delta + \phi \frac{1}{|N(s)|} \sum_{s' \in N(s)} y(s'),\ \sigma^2 \right)$$

Simultaneous Autogressve (SAR)

Using

$$y(s) = \phi \frac{1}{|N(s)|} \sum_{s' \in N(s)} y(s') + N(0, \sigma^2)$$

we want to find the distribution of $\boldsymbol{y} = \Big(y(s_1),\ y(s_2),\ \ldots,\ y(s_n) \Big)^t$.

First we can define a weight matrix $\boldsymbol{W}$ where

$$\{\boldsymbol{W}\}_{ij} = \begin{cases} 1/|N(s_i)| & \text{if } j \in N(s_i) \\ 0 & \text{otherwise} \end{cases}$$

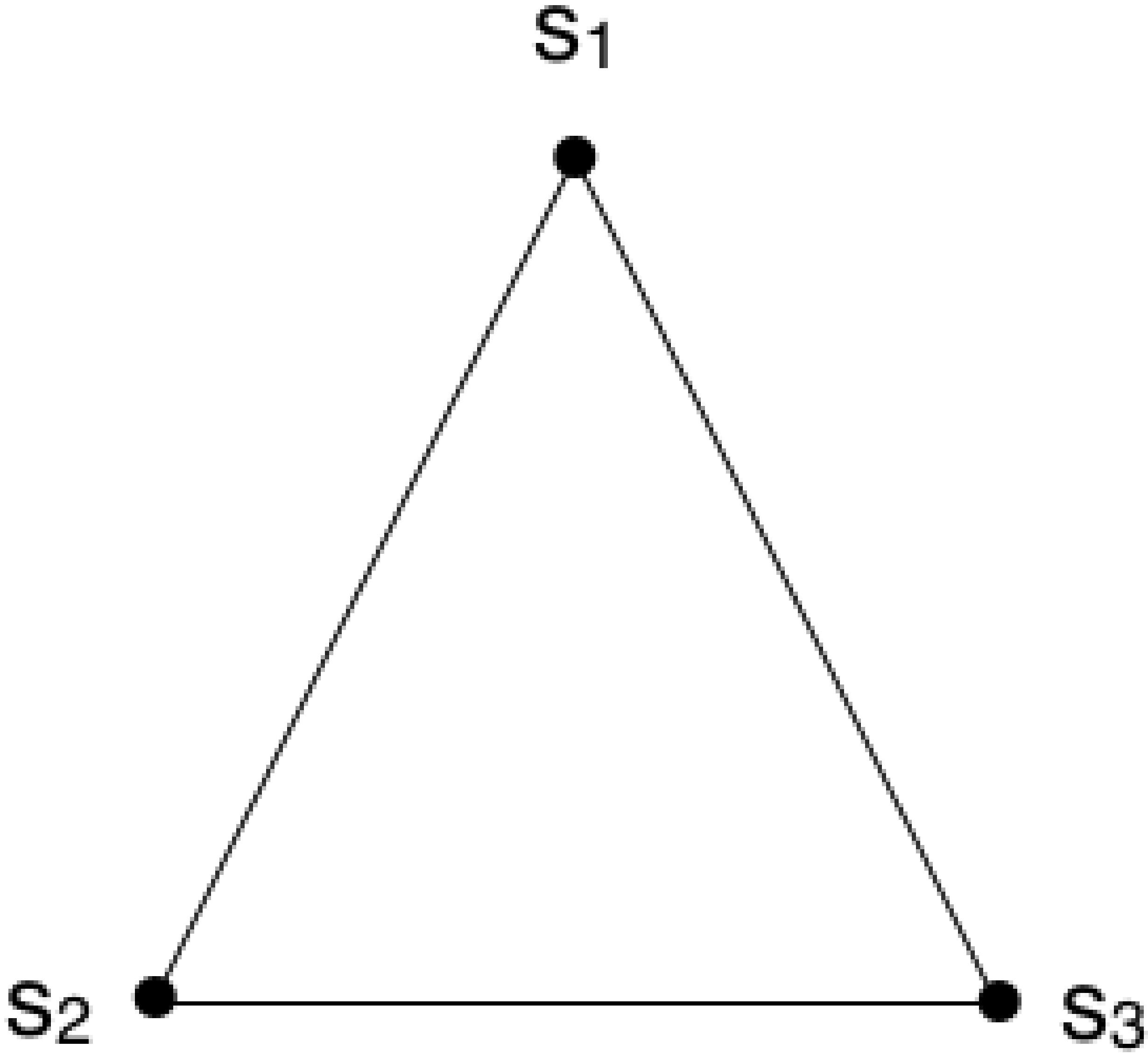
then we can write $\boldsymbol{y}$ as follows,

$$\boldsymbol{y} = \phi\, \boldsymbol{W}\, \boldsymbol{y} + \boldsymbol{\epsilon}$$

where

$$\boldsymbol{\epsilon} \sim N(0, \sigma^2\, \boldsymbol{I})$$

A toy example



Back to SAR

$$\boldsymbol{y} = \phi \, \boldsymbol{W} \, \boldsymbol{y} + \boldsymbol{\epsilon}$$

Conditional Autogressve (CAR)

This is a bit trickier, in the case of the temporal AR process we actually went from joint distribution $\longrightarrow$ conditional distributions (which we were then able to simplify).

Since we don't have a natural ordering we can't get away with this (at least not easily).

Going the other way, conditional distributions $\longrightarrow$ joint distribution is difficult because it is possible to specify conditional distributions that lead to an improper joint distribution.

Brooks' Lemma

For sets of observations $\boldsymbol{x}$ and $\boldsymbol{y}$ where $p(x) > 0 \;\; \forall \; x \in \boldsymbol{x}$ and $p(y) > 0 \;\; \forall \; y \in \boldsymbol{y}$ then

$$\frac{p(\boldsymbol{y})}{p(\boldsymbol{x})} = \prod_{i=1}^{n} \frac{p(y_i \mid y_1, \ldots, y_{i-1}, x_{i+1}, \ldots, x_n)}{p(x_i \mid y_1, \ldots, y_{i-1}, x_{i+1}, \ldots, x_n)}$$

$$= \prod_{i=1}^{n} \frac{p(y_i \mid x_1, \ldots, x_{i-1}, y_{i+1}, \ldots, y_n)}{p(x_i \mid x_1, \ldots, x_{i-1}, y_{i+1}, \ldots, y_n)}$$

A simplified example

Let $\boldsymbol{y} = (y_1, y_2)$ and $\boldsymbol{x} = (x_1, x_2)$ then we can derive Brook's Lemma for this case,

$$
\begin{aligned}
p(y_1, y_2) &= p(y_1 \mid y_2)p(y_2) \\
&= p(y_1 \mid y_2)\frac{p(y_2 \mid x_1)}{p(x_1 \mid y_2)}p(x_1) \\
&= \frac{p(y_1 \mid y_2)}{p(x_1 \mid y_2)}p(y_2 \mid x_1)\, p(x_1) \\
&= \frac{p(y_1 \mid y_2)}{p(x_1 \mid y_2)}p(y_2 \mid x_1)\, p(x_1) \left(\frac{p(x_2 \mid x_1)}{p(x_2 \mid x_1)}\right) \\
&= \frac{p(y_1 \mid y_2)}{p(x_1 \mid y_2)} \frac{p(y_2 \mid x_1)}{p(x_2 \mid x_1)}\, p(x_1, x_2)
\end{aligned}
$$

This is just a derivation of Brook's lemma for $\boldsymbol{y} = (y_1, y_2)$ and $\boldsymbol{x} = (x_1, x_2)$.

$$p(y_1, y_2) = \frac{p(y_1 \mid y_2)}{p(x_1 \mid y_2)} \frac{p(y_2 \mid x_1)}{p(x_2 \mid x_1)} p(x_1, x_2)$$

$$\frac{p(y_1, y_2)}{p(x_1, x_2)} = \frac{p(y_1 \mid y_2)}{p(x_1 \mid y_2)} \frac{p(y_2 \mid x_1)}{p(x_2 \mid x_1)}$$

From which we can generalize for $n = 3$,

$$\frac{p(y_1, y_2, y_3)}{p(x_1, x_2, x_3)} = \frac{p(y_1 \mid y_2, y_3)}{p(x_1 \mid y_2, y_3)} \frac{p(y_2 \mid x_1, y_3)}{p(x_2 \mid x_1, y_3} \frac{p(y_3 \mid x_1, x_2)}{p(x_3 \mid x_1, x_2)}$$

and so on.

Utility?

Lets repeat that last example but consider the case where $\boldsymbol{y} = (y_1, y_2)$ but now we let $\boldsymbol{x} = (y_1 = 0, y_2 = 0)$

$$\frac{p(y_1, y_2)}{p(x_1, x_2)} = \frac{p(y_1, y_2)}{p(y_1 = 0, y_2 = 0)}$$

$$p(y_1, y_2) = \frac{p(y_1 \mid y_2)}{p(y_1 = 0 \mid y_2)} \frac{p(y_2 \mid y_1 = 0)}{p(y_2 = 0 \mid y_1 = 0)} p(y_1 = 0, y_2 = 0)$$

$$\begin{aligned} p(y_1, y_2) &\propto \frac{p(y_1 \mid y_2)\, p(y_2 \mid y_1 = 0)}{p(y_1 = 0 \mid y_2)} \\ &\propto \frac{p(y_2 \mid y_1)\, p(y_1 \mid y_2 = 0)}{p(y_2 = 0 \mid y_1)} \end{aligned}$$

As applied to a simple CAR

$$y(s_1) \mid y(s_2) \sim N(\phi W_{12}\, y(s_2),\, \sigma^2)$$

$$y(s_2) \mid y(s_1) \sim N(\phi W_{21}\, y(s_1),\, \sigma^2)$$

$$p\Big(y(s_1), y(s_2)\Big) \propto \frac{p\Big(y(s_1) \mid y(s_2)\Big)\; p\Big(y(s_2) \mid y(s_1) = 0\Big)}{p\Big(y(s_1) = 0 \mid y(s_2)\Big)}$$

$$\propto \frac{\exp\left(-\frac{1}{2\sigma^2}(y(s_1) - \phi\, W_{12}\, y(s_2))^2\right) \exp\left(-\frac{1}{2\sigma^2}(y(s_2) - \phi\, W_{21}\; 0)^2\right)}{\exp\left(-\frac{1}{2\sigma^2}(0 - \phi W_{12} y(s_2))^2\right)}$$

$$\propto \exp\left(-\frac{1}{2\sigma^2}\left((y(s_1) - \phi\, W_{12}\, y(s_2))^2 + y(s_2)^2 - (\phi W_{21} y(s_2))^2\right)\right)$$

$$\propto \exp\Big(-\frac{1}{2\sigma^2}\big(y(s_1)^2 - \phi\, W_{12}\, y(s_1)\, y(s_2) - \phi\, W_{21}\, y(s_1)\, y(s_2) + y(s_2)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2}(\boldsymbol{y} - 0)\begin{pmatrix} 1 & -\phi W_{12} \\ -\phi W_{21} & 1 \end{pmatrix}(\boldsymbol{y} - 0)^t\right)$$

Implications for $\boldsymbol{y}$

$$\boldsymbol{\mu} = 0$$

$$\begin{aligned}
\boldsymbol{\Sigma}^{-1} &= \frac{1}{\sigma^2}\begin{pmatrix} 1 & -\phi W_{12} \\ -\phi W_{21} & 1 \end{pmatrix} \\
&= \frac{1}{\sigma^2}(\boldsymbol{I} - \phi\, \boldsymbol{W})
\end{aligned}$$

$$\boldsymbol{\Sigma} = \sigma^2(\boldsymbol{I} - \phi\, \boldsymbol{W})^{-1}$$

we can then conclude that for $\boldsymbol{y} = (y(s_1),\ y(s_2))^t$,

$$\boldsymbol{y} \sim N\left(\mathbf{0},\ \sigma^2(\boldsymbol{I} - \phi\, \boldsymbol{W})^{-1}\right)$$

which generalizes for all mean 0 CAR models.

General Proof

Let $\boldsymbol{y} = (y(s_1), \ldots, y(s_n))$ and $\boldsymbol{0} = (y(s_1) = 0, \ldots, y(s_n) = 0)$ then by Brook's lemma,

$$\frac{p(\boldsymbol{y})}{p(\boldsymbol{0})} = \prod_{i=1}^{n} \frac{p(y_i \mid y_1, \ldots, y_{i-1}, 0_{i+1}, \ldots, 0_n)}{p(0_i \mid y_1, \ldots, y_{i-1}, 0_{i+1}, \ldots, 0_n)} = \prod_{i=1}^{n} \frac{\exp\left(-\frac{1}{2\sigma^2}\left(y_i - \phi \sum_{j<i} W_{ij}\, y_j - \phi \sum_{j>i} 0_j\right)^2\right)}{\exp\left(-\frac{1}{2\sigma^2}\left(0_i - \phi \sum_{j<i} W_{ij}\, y_j - \phi \sum_{j>i} 0_j\right)^2\right)}$$

$$= \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} \left(y_i - \phi \sum_{j<i} W_{ij}\, y_j\right)^2 + \frac{1}{2\sigma^2} \sum_{i=1}^{n} \left(\phi \sum_{j<i} W_{ij}\, y_j\right)^2\right)$$

$$= \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} y_i^2 - 2\phi y_i \sum_{j<i} W_{ij}\, y_j\right)$$

$$= \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} y_i^2 - \phi \sum_{i=1}^{n} \sum_{j=1}^{n} y_i\, W_{ij}\, y_j\right) \quad \left(\text{if } W_{ij} = W_{ji}\right)$$

$$= \exp\left(-\frac{1}{2\sigma^2} (\boldsymbol{y} - 0)^t (\boldsymbol{I} - \phi \boldsymbol{W})(\boldsymbol{y} - 0)\right)$$

CAR vs SAR

- Simultaneous Autogressve (SAR)

$$y(s) = \phi \sum_{s'} W_{s\,s'}\, y(s') + \epsilon$$

$$\boldsymbol{y} \sim N(0,\ \sigma^2\,((\boldsymbol{I} - \phi \boldsymbol{W})^{-1})((\boldsymbol{I} - \phi \boldsymbol{W})^{-1})^t)$$

- Conditional Autoregressive (CAR)

$$y(s) \mid y(-s) \sim N\left(\sum_{s'} W_{s\,s'}\, y(s'),\ \sigma^2\right)$$

$$\boldsymbol{y} \sim N(0,\ \sigma^2\,(\boldsymbol{I} - \phi \boldsymbol{W})^{-1})$$

Generalizations

- Adopting different weight matrices $(\boldsymbol{W})$
  - Between SAR and CAR model we move to a generic weight matrix definition (beyond average of nearest neighbors)
  - In time we varied p in the $\mathrm{AR(p)}$ model, in space we adjust the weight matrix.
  - In general having a symmetric W is helpful, but not required

- More complex Variance (beyond $\sigma^2\ \mathrm{I}$)
  - σ^2 can be a vector (differences between areal locations)
  - i.e. since areal data tends to be aggregated - adjust variance based on sample size
  - i.e. scale based on the number of neighbors

Some specific generalizations

Generally speaking we want to work with a scaled / normalized version of the weight matrix,

$$W_{ij}/W_{i\cdot}$$

When W is derived from an adjacency matrix, $\boldsymbol{A}$, we can express this as

$$\boldsymbol{W} = \boldsymbol{D}^{-1}\boldsymbol{A}$$

where $\boldsymbol{D}^{-1} = \mathrm{diag}(1/|N(s_i)|)$.

We can also allow σ^2 to vary between locations, we can define this as $\boldsymbol{D}_{\sigma^2} = \mathrm{diag}(\sigma_i^2)$ and most often we use

$$\boldsymbol{D}_{\sigma^2}^{-1} = \mathrm{diag}\left(\frac{\sigma^2}{|N(s_i)|}\right) = \sigma^2\boldsymbol{D}^{-1}.$$

Revised SAR Model

- Formula Model

$$y(s_i) = X_{i\cdot}\beta + \phi \sum_{j=1}^{n} D_{jj}^{-1} A_{ij} \left(y(s_j) - X_{j\cdot}\beta \right) + \epsilon_i$$

$$\boldsymbol{\epsilon} \sim N(\mathbf{0}, \boldsymbol{D}_{\sigma^2}^{-1}) = N(\mathbf{0}, \sigma^2 \boldsymbol{D}^{-1})$$

- Joint Model

$$\boldsymbol{y} = \boldsymbol{X\beta} + \phi \boldsymbol{D}^{-1} \boldsymbol{A} \left(\boldsymbol{y} - \boldsymbol{X\beta} \right) + \boldsymbol{\epsilon}$$

$$\boldsymbol{y} \sim N\left(\boldsymbol{X\beta}, (\boldsymbol{I} - \phi \boldsymbol{D}^{-1} \boldsymbol{A})^{-1} \sigma^2 \boldsymbol{D}^{-1} \left((\boldsymbol{I} - \phi \boldsymbol{D}^{-1} \boldsymbol{A})^{-1} \right)^t \right)$$

Revised CAR Model

- Conditional Model

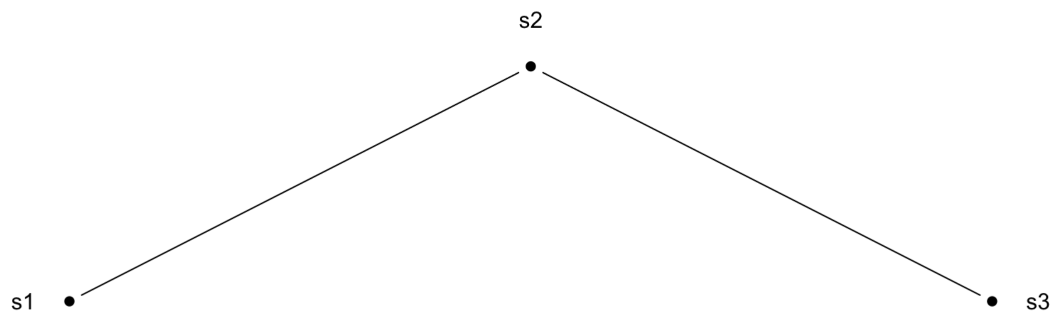
$$y(s_i) \mid y_{-s_i} \sim N\left(X_{i\cdot}\beta + \phi \sum_{j=1}^{n} \frac{W_{ij}}{D_{ii}} \left(y(s_j) - X_{j\cdot}\beta\right),\ \sigma^2 D_{ii}^{-1} \right)$$

- Joint Model

$$\boldsymbol{y} \sim N(\boldsymbol{X\beta},\ \Sigma_{CAR})$$

$$\begin{aligned}\Sigma_{CAR} &= (\boldsymbol{D}_\sigma\,(\boldsymbol{I} - \phi \boldsymbol{D}^{-1}\boldsymbol{A}))^{-1} \\ &= (1/\sigma^2 \boldsymbol{D}\,(\boldsymbol{I} - \phi \boldsymbol{D}^{-1}\boldsymbol{A}))^{-1} \\ &= (1/\sigma^2 (\boldsymbol{D} - \phi \boldsymbol{A}))^{-1} \\ &= \sigma^2 (\boldsymbol{D} - \phi \boldsymbol{A})^{-1}\end{aligned}$$

Toy CAR Example



$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \boldsymbol{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

When does Σ exist?

```r
check_sigma = function(phi) {
  Sigma_inv = matrix(c(1,-phi,0,-phi,2,-phi,0,-phi,1), ncol=3, byrow=TRUE)
  solve(Sigma_inv)
}

check_sigma(phi=0)
```

```
     [,1] [,2] [,3]
[1,]    1  0.0    0
[2,]    0  0.5    0
[3,]    0  0.0    1
```

```r
check_sigma(phi=0.5)
```

```
          [,1]      [,2]      [,3]
[1,] 1.1666667 0.3333333 0.1666667
[2,] 0.3333333 0.6666667 0.3333333
[3,] 0.1666667 0.3333333 1.1666667
```

```r
check_sigma(phi=-0.6)
```

```
         [,1]     [,2]     [,3]
[1,]  1.28125 -0.46875  0.28125
[2,] -0.46875  0.78125 -0.46875
[3,]  0.28125 -0.46875  1.28125
```

```r
check_sigma(phi=1)
```

```
Error in solve.default(Sigma_inv): Lapack routine dgesv: system is exactly singular: U[3,3] = 0
```

```r
check_sigma(phi=-1)
```

```
Error in solve.default(Sigma_inv): Lapack routine dgesv: system is exactly singular: U[3,3] = 0
```

```r
check_sigma(phi=1.2)
```

```
           [,1]      [,2]       [,3]
[1,] -0.6363636 -1.363636 -1.6363636
[2,] -1.3636364 -1.136364 -1.3636364
[3,] -1.6363636 -1.363636 -0.6363636
```

```r
check_sigma(phi=-1.2)
```

```
           [,1]      [,2]       [,3]
[1,] -0.6363636  1.363636 -1.6363636
[2,]  1.3636364 -1.136364  1.3636364
[3,] -1.6363636  1.363636 -0.6363636
```

When is Σ positive semidefinite?

```r
check_sigma_pd = function(phi) {
  Sigma_inv = matrix(c(1,-phi,0,-phi,2,-phi,0,-phi,1), ncol=3, byrow=TRUE)
  chol(solve(Sigma_inv))
}

check_sigma_pd(phi=0)
```

```
     [,1]      [,2] [,3]
[1,]    1 0.0000000    0
[2,]    0 0.7071068    0
[3,]    0 0.0000000    1
```

```r
check_sigma_pd(phi=0.5)
```

```
         [,1]      [,2]      [,3]
[1,] 1.080123 0.3086067 0.1543033
[2,] 0.000000 0.7559289 0.3779645
[3,] 0.000000 0.0000000 1.0000000
```

```r
check_sigma_pd(phi=-0.6)
```

```
         [,1]       [,2]       [,3]
[1,] 1.131923 -0.4141182  0.2484709
[2,] 0.000000  0.7808688 -0.4685213
[3,] 0.000000  0.0000000  1.0000000
```

```r
check_sigma_pd(phi=1)
```

```
Error in solve.default(Sigma_inv): Lapack routine dgesv: system is exactly singular: U[3,3] = 0
```

```r
check_sigma_pd(phi=-1)
```

```
Error in solve.default(Sigma_inv): Lapack routine dgesv: system is exactly singular: U[3,3] = 0
```

```r
check_sigma_pd(phi=1.2)
```

```
Error in chol.default(solve(Sigma_inv)): the leading minor of order 1 is not positive definite
```

```r
check_sigma_pd(phi=-1.2)
```

```
Error in chol.default(solve(Sigma_inv)): the leading minor of order 1 is not positive definite
```

Conclusions

Generally speaking just like the AR(1) model for time series we require that $|\phi| < 1$ for the CAR model to be proper.

These results for ϕ also apply in the context where σ_i^2 is constant across locations, i.e.

$$\boldsymbol{\Sigma} = \left(\sigma^2 \left(\boldsymbol{I} - \phi \boldsymbol{D}^{-1} \boldsymbol{A}\right)\right)^{-1}$$

As a side note, the special case where $\phi = 1$ is known as an intrinsic autoregressive (IAR) model and they are popular as an *improper* prior for spatial random effects. An additional sum constraint is necessary for identifiability

$$\sum_{i=1}^{n} y(s_i) = 0$$